Peak to Average Power Control via Tone Reservation in General Orthonormal Transmission Systems

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***Abstract*—In this paper we study the tone reservation method for reducing the peak to average power ratio (PAPR) in general orthonormal transmission systems. We prove that strong solvability, where the peak value of the transmit signal has to be bounded by a constant times the energy of the information symbols, is equivalent to weak solvability, where the peak value of the transmit signal has to be only bounded. Further, we show that in the case where the PAPR problem is not weakly solvable, almost all information sequences lead to an unbounded transmit signal.**

***Index Terms*—Orthonormal transmission system, peak to average power ratio, tone reservation, weak solvability, strong solvability**

I. Introduction

For communication systems that target high spectral efficiencies, multi waveform transmission methods, such as OFDM and generalizations, are considered. Many of these techniques are also a current research topic in the development of 5G. One drawback of multi waveform schemes are large peak values of the transmit signal, which might occur, depending on the information symbols to be transmitted. These high signal values can be a problem for the hardware, in particular the power amplifier.

A central factor in the design of future communication systems is their efficiency in terms of energy consumption during operation. Besides ecological considerations, a low energy consumption is essential for the cost effectiveness of operating a network. In general, the ambitions to reduce the energy consumption in electrical systems led to the idea of "green IT" several years ago. In view of the exponential increase of the data that is transported in communication networks, the importance of these questions will even increase. However, many questions that are related to energy efficient hardware design turned out to be challenging and only few of the original goals have been achieved.

Our considerations are related to the efficiency of the power amplifier in wireless communication systems. In theory one often assumes an ideal amplifier characteristic, i.e., a linear input-output behavior. In practice, however, such a behavior cannot be realized at a reasonable cost. Outside an operating range, where the amplifier can be considered to be linear, the gain is reduced and the amplifier saturates. Large signal values at the input can overload the amplifier, leading to signal distortion and out-of-band radiation [1]. Consequently, the design of suitable waveform transmission schemes with small peak to average power ratios (PAPRs) is an important task. For the design of suitable waveforms, it is further vital to consider the physical properties of the transmission channel as well as the possibilities of the hardware for implementation.

Classical transmission schemes, such as orthogonal frequency division multiplexing (OFDM) and code division multiple access (CDMA), suffer from large PAPRs [2]–[5]. For future communication systems other, more general waveform transmission schemes are discussed [6]. Large PAPR values, however, are not specific to OFDM and CDMA systems, but rather occur for arbitrary multi waveform schemes based on bounded orthonormal systems (ONS). It is well known that the PAPR of such signals can be as large as $\sqrt{N}$, where N denotes the number of carriers [7].

In order to reduce the PAPR, several methods have been proposed [8]–[12], among them the popular tone reservation method [13]–[17], which we consider in this paper. In this method, the set of available carriers is partitioned into two sets, the first of which is used to carry the information (information set), and the second of which to reduce the PAPR (compensation set). In the tone reservation method only the transmitter has to find the suitable compensation symbols. The information set is known in advance to both the transmitter and receiver, and therefore no additional coordination or information exchange between transmitter and receiver is needed. The receiver simply disregards the compensation symbols and uses only the information symbols.

In the tone reservation method not the entire available energy is used for the transmission of information, instead a certain fraction is used for compensation, i.e., reduction of the PAPR of the transmit signal. The reduced PAPR of the transmit signal makes it possible to use more efficient power amplifiers that save energy. Hence, from an energy perspective, there is a trade-off, and to date a thorough quantitative analysis of this trade-off is missing. A first step towards the development of such an analysis, is a better theoretical understanding of the tone reservation method.

Tone reservation is an elegant procedure and easy to define. The practical implementation however is difficult. Little of the available results are analytic in nature, and, to the best of our knowledge, there exists no efficient algorithms for the calculation of the compensation set for an arbitrary number of carriers. Central questions in the context of tone reservation are: What is the best possible reduction of the PAPR? What

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is the optimal information set that achieves this reduction, and how can it be found? What is the general form of the information set? There has been little theoretical work done in this area and so far none of these questions could be fully answered.

In order to tackle these questions it is important to understand the structure of the problem. Two different solvability concepts have been proposed: strong and weak solvability. The concept of weak solvability was first discussed in Section 4.7 of the chapter "Mathematics of signal design for communication systems" in [18]. However, when [18] was published, it was unclear how the PAPR problem behaves for the concept of weak solvability. In [19] it was shown for OFDM that if the PAPR problem is not weakly solvable, then the set of information sequences for which the transmit signal is unbounded is a residual set, regardless of the choice of $b \in \ell^2(\mathcal{K})$. Further, it was proved in [19] that for OFDM both concepts—weak and strong solvability of the PAPR problem—are equivalent. In this paper, we generalize this result to arbitrary complete ONSs. On the technical side, this generalization is far from trivial. The proof in [19] exploited specific properties of the system of exponentials that is used in OFDM, and therefore does not work for arbitrary complete ONSs. Hence, in the present paper, we need to use a completely different approach for the proof. The generalization to arbitrary complete ONSs is a significant step, because the development of orthogonal transmission schemes is still ongoing, and other waveforms than those employed in OFDM are actively considered in modern standards, such as 5G [20].

In Section III we will explain the tone reservation method in more detail, and in Section IV we introduce the concepts of weak and strong solvability of the PAPR problem. Then in Section V, we present our main result, the equivalence of weak and strong solvability for arbitrary bounded complete ONSs. In the Section VI, we apply the findings and show that if the PAPR problem is not weakly solvable, then for a large class of transmit signals the PAPR cannot be controlled, i.e., a compensation is not possible. This is the generic behavior, because it occurs for "almost all" signals.

II. Notation

By $L^p[0,1]$, $1 \le p \le \infty$, we denote the usual L^p-spaces on the interval $[0,1]$, equipped with the norm $\|\cdot\|_{L^p[0,1]}$. For an index set $\mathcal{I} \subset \mathbb{Z}$, we denote by $\ell^2(\mathcal{I})$ the set of all square summable sequences $c = \{c_k\}_{k\in\mathcal{I}}$ indexed by $\mathcal{I}$. The norm is given by $\|c\|_{\ell^2(\mathcal{I})} = (\sum_{k\in\mathcal{I}}|c_k|^2)^{1/2}$. By $|A|$ we denote the cardinality of a set A, and by $\overline{z}$ the complex conjugate of a complex number z.

The Rademacher functions r_n, $n \in \mathbb{N}$, on $[0,1]$ are defined by $r_n(t) = \operatorname{sgn}[\sin(\pi 2^n t)]$, where sgn denotes the signum function with the convention $\operatorname{sgn}(0) = -1$. The Walsh functions w_n, $n \in \mathbb{N}$, on $[0,1]$ are defined by

$$w_1(t) = 1$$

and

$$w_{2^k+m}(t) = r_{k+1}(t) w_m(t)$$

for $k = 0, 1, 2, \ldots$ and $m = 1, 2, \ldots, 2^k$. Note that we use an indexing of the Walsh functions that starts with 1. The Walsh functions $\{w_n\}_{n\in\mathbb{N}}$ form an orthonormal basis for $L^2[0,1]$. For further details about the Walsh function, see for example [21].

Further, we need the following concepts from metric spaces [22]. A subset M of a metric space X is said to be nowhere dense in X if the closure $[M]$ does not contain a non-empty open set of X. M is said to be meager (or of the first category) if M is the countable union of sets each of which is nowhere dense in X. M is said to be nonmeager (or of the second category) if it is not meager. The complement of a meager set is called a residual set. According to Baire's theorem [22], in a complete metric space any residual set is dense and nonmeager. Residual sets may be considered as "big" and meager sets as "small". Two properties that shows the richness of residual sets are the following: the countable intersection of residual sets is always a residual set; and any superset of a residual set is a residual set. The small size of meager sets is illustrated by the next two properties: The countable union of meager sets is always a meager set; and any subset of a meager set is a meager set. A rough analog in probability theory are sets having probability one and sets having zero probability.

III. Problem Formulation and Tone Reservation

A. PAPR for Wave Form Transmission

We start our discussion with one-dimensional time signals, as they appear in applications. Please note that our main results, presented in Section V, hold in greater generality for measure spaces, including multidimensional signals.

Without loss of generality, we can restrict ourselves to signals defined on the interval $[0,1]$. Signals with other duration can be simply scaled to be concentrated on $[0,1]$. For a signal s, we define

$$\mathrm{PAPR}(s) = \frac{\|s\|_{L^\infty[0,1]}}{\|s\|_{L^2[0,1]}},$$

i.e., the PAPR is the ratio between the peak value of the signal and the square root of the power of the signal. Since $\|s\|_{L^2[0,1]} \le \|s\|_{L^\infty[0,1]}$, we always have

$$\mathrm{PAPR}(s) \ge 1. \tag{1}$$

Note that the PAPR is usually defined as the square of this value. This however, from a mathematical point of view, makes no difference for the results in this paper.

In the case of an orthogonal transmission scheme, using the ONS $\{\phi_k\}_{k\in\mathcal{I}} \subset L^2[0,1]$, the PAPR of the transmit signal

$$s(t) = \sum_{k\in\mathcal{I}} c_k \phi_k(t), \quad t \in [0,1],$$

with coefficients $c = \{c_k\}_{k\in\mathcal{I}}$, is given by

$$\mathrm{PAPR}(s) = \frac{\|\sum_{k\in\mathcal{I}} c_k \phi_k\|_{L^\infty[0,1]}}{\|c\|_{\ell^2(\mathcal{I})}},$$

because $\{\phi_k\}_{k\in\mathcal{I}}$ is a ONS, which implies that $\|s\|_{L^2[0,1]} = \|c\|_{\ell^2(\mathcal{I})}$.

For an orthogonal transmission scheme, the peak value of the signal s, and hence the PAPR, can become large, as the

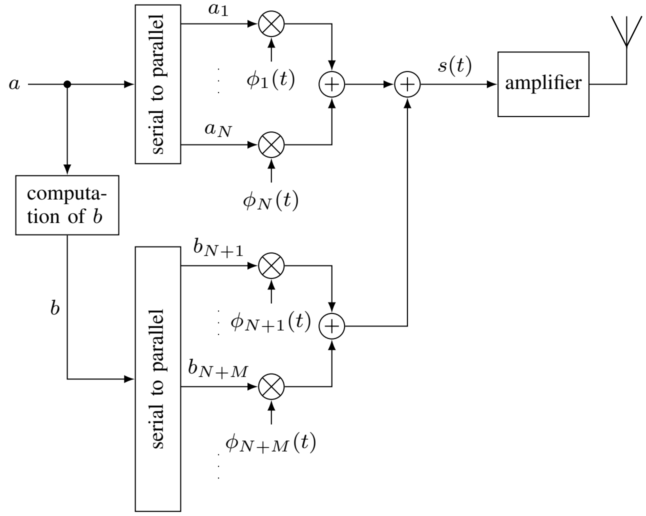

Fig. 1. Block diagram of a general orthogonal waveform transmission scheme with tone reservation. In this example $\{\phi_n\}_{n\in\mathbb{N}}$ is a complete ONS, and we have $\mathcal{I} = \mathbb{N}$, $\mathcal{K} = \{1, \dots, N\}$, and $\mathcal{K}^{\complement} = \mathbb{N} \setminus \mathcal{K}$.

following result shows. Given any system $\{\phi_n\}_{n=1}^N$ of N orthonormal functions in $L^2[0,1]$, then there exist a sequence $\{c_n\}_{n=1}^N \subset \mathbb{C}$ of coefficients with $\sum_{n=1}^N |c_n|^2 = 1$, such that $\|\sum_{n=1}^N c_n\phi_n\|_{L^\infty[0,1]} \geq \sqrt{N}$ [7].

For the OFDM case, where the complete ONS consists of complex exponentials $\{\mathrm{e}^{ik\,\cdot\,2\pi}\}_{k\in\mathbb{Z}}$ and the CDMA case, where the complete ONS is given by the Walsh functions $\{w_n\}_{n\in\mathbb{N}}$, this can be easily seen. For the sequence

$$c_n = \begin{cases} \frac{1}{\sqrt{N}}, & 1 \leq n \leq N, \\ 0, & \text{otherwise,} \end{cases}$$

we clearly have $\sum_{n=1}^N |c_n|^2 = 1$. In the OFDM case we have

$$\left\| \sum_{n=1}^N c_n\, \mathrm{e}^{int2\pi} \right\|_{L^\infty[0,1]} = \frac{1}{\sqrt{N}} \max_{t\in[0,1]} \left| \sum_{n=1}^N \mathrm{e}^{int2\pi} \right| = \sqrt{N},$$

and in the Walsh case

$$\left\| \sum_{n=1}^N c_n w_n \right\|_{L^\infty[0,1]} = \frac{1}{\sqrt{N}} \left\| \sum_{n=1}^N w_n \right\|_{L^\infty[0,1]} = \sqrt{N},$$

because $\sum_{n=1}^N w_n(t) = N$ for all $t \in (0, 1/N)$.

The significance of tone reservation, which is illustrated in Fig. 1, has already been discussed in the introduction. Next, we introduce the method more precisely. Let $\{\phi_k\}_{k\in\mathcal{I}}$ be an ONS in $L^2[0,1]$. We additionally assume that $\|\phi_k\|_\infty < \infty$, $k \in \mathcal{I}$, i.e., we consider the practically relevant case of bounded functions. In the tone reservation method, the index set $\mathcal{I}$ is partitioned in two disjoint sets, the information set $\mathcal{K}$ and the compensation set $\mathcal{K}^{\complement}$. The set $\mathcal{K}$ is used to carry the information and the set $\mathcal{K}^{\complement}$ to reduce he PAPR. Note that the set $\mathcal{K}$ can be finite or infinite. For a given information sequence $a = \{a_k\}_{k\in\mathcal{K}} \in \ell^2(\mathcal{K})$, the goal is to find a compensation sequence $b = \{a_k\}_{k\in\mathcal{K}^{\complement}} \in \ell^2(\mathcal{K}^{\complement})$ such that the peak value of the transmit signal

$$s(t) = \underbrace{\sum_{k\in\mathcal{K}} a_k\phi_k(t)}_{=:A(t)} + \underbrace{\sum_{k\in\mathcal{K}^{\complement}} b_k\phi_k(t)}_{=:B(t)}, \quad t \in [0,1],$$

is as small as possible. $A(t)$ denotes the signal part which contains the information and $B(t)$ the part which is used to reduce the PAPR.

Note that we allow infinitely many carriers to be used for the compensation of the PAPR. This is also of practical interest, since the solvability of the PAPR problem in this setting is a necessary condition for the solvability of the PAPR problem in the setting with finitely many carriers.

B. General Complete ONS

In the remainder of this paper we generalize the problem to ONSs on measure spaces. This includes multidimensional signals but also one-dimensional signals as discussed in the previous section.

Let Ω be a nonempty set, $\mathfrak{A}$ a σ-algebra on the set Ω, and μ a measure on $(\Omega, \mathfrak{A})$, such that $(\Omega, \mathfrak{A}, \mu)$ be a separable measure space with $\mu(\Omega) < \infty$. Without loss of generality we will assume that μ is a probability measure, i.e., that $\mu(\Omega) = 1$, because this avoids unpleasant normalization constants. For $1 \leq p \leq \infty$, we set

$$L^p(\mu) = \left\{ f \text{ measurable} \colon \|f\|_{L^p(\mu)} < \infty \right\},$$

where

$$\|f\|_{L^p(\mu)} = \left(\int_\Omega |f|^p \,\mathrm{d}\mu \right)^{\frac{1}{p}}$$

for $1 \leq p < \infty$, and

$$\|f\|_{L^\infty(\mu)} = \operatorname*{ess\,sup}_{t\in\Omega} |f(t)|$$

for $p = \infty$. By

$$\operatorname*{ess\,sup}_{t\in\Omega} g(t) = \inf\{M \in \mathbb{R} \colon \mu(\{t \in \Omega \colon g(t) > M\}) = 0\}$$

we denote the essential supremum of a measurable function $g \colon \Omega \to \mathbb{R}$. As usual, we identify two functions that are equal μ-almost everywhere (a.e.). $L^2(\mu)$ is a Hilbert space. Since $L^2(\mu)$ is separable, and $L^2(\mu)$ is dense in $L^1(\mu)$, it follows that $L^1(\mu)$ is separable.

Let $\{\phi_n\}_{n\in\mathbb{N}}$ be a complete ONS in $L^2(\mu)$. Then, for all $f \in L^2(\mu)$, we have

$$\lim_{N\to\infty} \int_\Omega \left| f - \sum_{n=1}^N a_n(f)\phi_n \right|^2 \mathrm{d}\mu = 0,$$

where

$$a_n(f) = \int_\Omega f\overline{\phi_n} \,\mathrm{d}\mu,$$

as well as

$$\sum_{n=1}^\infty |a_n(f)|^2 = \int_\Omega |f|^2 \,\mathrm{d}\mu.$$

IV. Strong and Weak Solvability

Next we introduce two solvability concepts for the tone reservation method. The difference of the two concepts lies in the way how we control the peak value, i.e. the L^∞ norm, of the transmit signal. The first solvability concept, which was formally introduced in [16], [23], is strong solvability.

Definition 1 (Strong solvability of the PAPR problem)**.** For an ONS $\{\phi_k\}_{k\in\mathcal{I}}$ in $L^2(\mu)$ and a set $\mathcal{K}\subset\mathcal{I}$, we say that the PAPR problem is strongly solvable with finite constant C_{EX}, if for all $a\in\ell^2(\mathcal{K})$ there exists a $b\in\ell^2(\mathcal{K}^\complement)$ such that

$$\left\|\sum_{k\in\mathcal{K}} a_k\phi_k + \sum_{k\in\mathcal{K}^\complement} b_k\phi_k\right\|_{L^\infty(\mu)} \le C_{\text{EX}}\|a\|_{\ell^2(\mathcal{K})}. \tag{2}$$

We call the PAPR problem strongly solvable if it is strongly solvable for some finite constant C_{EX}.

If the PAPR problem is strongly solvable then the peak value of the transmit signal

$$s = \underbrace{\sum_{k\in\mathcal{K}} a_k\phi_k}_{=:A} + \underbrace{\sum_{k\in\mathcal{K}^\complement} b_k\phi_k}_{=:B}$$

can be controlled, i.e., bounded from above by $C_{\text{EX}}\|a\|_{\ell^2(\mathcal{K})}$, for all information sequences $a\in\ell^2(\mathcal{K})$. Note that we have

$$\|s\|^2_{L^2(\mu)} = \|a\|^2_{\ell^2(\mathcal{K})} + \|b\|^2_{\ell^2(\mathcal{K}^\complement)} \tag{3}$$

because Parseval's equality. It follows that

$$\begin{aligned}\|b\|^2_{\ell^2(\mathcal{K}^\complement)} &= \|s\|^2_{L^2(\mu)} - \|a\|^2_{\ell^2(\mathcal{K})}\\ &\le \|s\|^2_{L^\infty(\mu)} - \|a\|^2_{\ell^2(\mathcal{K})}\\ &\le C^2_{\text{EX}}\|a\|^2_{\ell^2(\mathcal{K})} - \|a\|^2_{\ell^2(\mathcal{K})}\\ &= (C^2_{\text{EX}}-1)\|a\|^2_{\ell^2(\mathcal{K})},\end{aligned} \tag{4}$$

where we used (2) in the last inequality. Thus, the energy of the compensation signal $\|B\|^2_{L^2(\mu)} = \|b\|^2_{\ell^2(\mathcal{K}^\complement)}$ is bounded from above by a constant factor $C^2_{\text{EX}}-1$ times the energy of the information signal $\|A\|^2_{L^2(\mu)} = \|a\|^2_{\ell^2(\mathcal{K}^\complement)}$. As a consequence, the constant C_{EX} controls not only the peak value of the transmit signal via (2), but also the energy of the compensation signal via (4). For practical applications it is desirable to have the minimum possible constant C_{EX} in (2) as small as possible. There is a natural lower bound on the minimum possible constant C_{EX}. Since

$$\text{PAPR}(s) = \frac{\|s\|_{L^\infty(\mu)}}{\|s\|_{L^2(\mu)}} \le \frac{C_{\text{EX}}\|a\|_{L^2(\mu)}}{\|a\|_{L^2(\mu)}} \le C_{\text{EX}}, \tag{5}$$

we obtain from (1) that $C_{\text{EX}}\ge 1$. Finding the minimum possible constant for a given ONS is a challenging task, which has been solved only for very few ONSs, e.g., the Walsh ONS (see Example 1). Note that the inequality (5) also shows that the constant C_{EX} is an upper bound on the PAPR value of s.

In [16], [18] an interesting alternative characterization of strong solvability was given. In this characterization the set

$$\mathfrak{F}^1(\mathcal{K}) = \left\{ f\in L^1(\mu) : f = \sum_{k\in\mathcal{K}} a_k\phi_k \text{ for some } \{a_k\}_{k\in\mathcal{K}}\subset\mathbb{C}\right\}$$

plays an important role. The theorem is as follows.

Theorem 1. *Let $\{\phi_n\}_{n\in\mathbb{N}}$ be a bounded complete ONS in $L^2(\mu)$, $\mathcal{K}\subset\mathbb{N}$, and $C_{EX}>0$. The PAPR problem is strongly solvable for $\{\phi_n\}_{n\in\mathbb{N}}$ and $\mathcal{K}$ with constant C_{EX} if and only if we have*

$$\|f\|_{L^2(\mu)} \le C_{EX}\|f\|_{L^1(\mu)} \tag{6}$$

for all $f\in\mathfrak{F}^1(\mathcal{K})$.

Remark 1. Theorem 1 was initially stated in [16], [18] for the Lebesgue measure on the interval $[0,1]$ and not for general measure spaces. However, a closer inspection of the proof shows that it can easily generalized to separable probability spaces $(\Omega,\mathfrak{A},\mu)$. For the "$\Leftarrow$" direction of the proof, the Hahn–Banach theorem is needed and the fact that the dual space of $L^1([0,1])$ is $L^\infty([0,1])$. Both are equally true in our probability space setting. The "$\Rightarrow$" direction follows immediately without changes.

It is easy to show there exist infinite sets $\mathcal{K}\subset\mathcal{I}$ for which the PAPR is strongly solvable.

We will give two examples next.

Example 1. For the Walsh ONS $\{w_n\}_{n\in\mathbb{N}}$ in $L^2[0,1]$ (CDMA case) we can use the information set $\mathcal{K}=\{2^l\}_{l\in\mathbb{N}\cup\{0\}}$. Then the PAPR problem is strongly solvable, and it can be shown that the optimal extension constant is $C_{\text{EX}}=\sqrt{2}$ [24].

Example 2. For the Fourier ONS $\{\mathrm{e}^{ik\,\cdot\,2\pi}\}_{k\in\mathbb{Z}}$ in $L^2[0,1]$ (OFDM case), the same information set $\mathcal{K}=\{2^l\}_{l\in\mathbb{N}\cup\{0\}}$ makes the PAPR problem strongly solvable. However, in this case the optimal extension constant is yet unknown.

For OFDM, using the complex exponentials, and CDMA, using the Walsh functions, the information sets $\mathcal{K}$ for which the PAPR is strongly solvable need to be sparse, similar to Examples 1 and 2, where the gaps grow larger and larger [19], [25]. In [16] the following result was proved for OFDM: If $\mathcal{K}\subset\mathbb{Z}$ is a set such that the PAPR is strongly solvable for $\mathcal{K}$ with some finite extension constant C_{EX} then we have

$$\lim_{N\to\infty}\frac{|\mathcal{K}\cap[-N,N]|}{2N+1}=0,$$

that is, the relative density of the information set in $[-N,N]$ needs to go to zero. A similar result was shown in [18], [23] for the Walsh system: If $\mathcal{K}\subset\mathbb{N}$ is a set such that the PAPR is strongly solvable for $\mathcal{K}$ with some finite extension constant C_{EX} then we have

$$\lim_{N\to\infty}\frac{|\mathcal{K}\cap[1,N]|}{N}=0.$$

This is true regardless of the specific value of the constant C_{EX}.

In view of the discouraging results about the density of the information set, one could ask if it is too restricting to require (2), i.e., the control of the peak value by a constant C_{EX} times the norm of a. Therefore, in [18] the concept of weak solvability was introduced.

Definition 2 (Weak solvability of the PAPR problem)**.** For an ONS $\{\phi_k\}_{k\in\mathcal{I}}$ in $L^2(\mu)$ and a set $\mathcal{K} \subset \mathcal{I}$, we say that the PAPR problem is weakly solvable if for all $a \in \ell^2(\mathcal{K})$ we have

$$\inf_{b\in\ell^2(\mathcal{K}^{\complement})} \left\| \sum_{k\in\mathcal{K}} a_k\phi_k + \sum_{k\in\mathcal{K}^{\complement}} b_k\phi_k \right\|_{L^\infty(\mu)} < \infty. \tag{7}$$

This is a weaker form of solvability compared to strong solvability, as stated in Definition 1. The peak value of the transmit signal

$$s = \underbrace{\sum_{k\in\mathcal{K}} a_k\phi_k}_{=:A} + \underbrace{\sum_{k\in\mathcal{K}^{\complement}} b_k\phi_k}_{=:B} \tag{8}$$

is only required to be bounded and not to be controlled by the norm of the sequence $a = \{a_k\}_{k\in\mathcal{K}}$ as in (2). We want to show next that this gives only a very weak form of control of the energy of the compensation signal. Since

$$\|b\|^2_{\ell^2(\mathcal{K}^{\complement})} \le \|s\|^2_{L^2(\mu)},$$

according to (3), and

$$\|s\|_{L^2(\mu)} \le \|s\|_{L^\infty(\mu)},$$

it follows that

$$\|B\|^2_{L^2(\mu)} = \|b\|^2_{\ell^2(\mathcal{K}^{\complement})} \le \|s\|^2_{L^\infty(\mu)}. \tag{9}$$

Hence, we see that the energy of the compensation signal is bounded if the PAPR reduction problem is weakly solvable. We further see that the finiteness of the energy of the compensation signal is a necessary condition for the finiteness of the PAPR: If the transmit signal s has a finite PAPR, then we necessarily have $\|s\|_{L^\infty(\mu)} < \infty$ according of the definition of the PAPR, and it follows from (9) that $\|B\|^2_{L^2(\mu)} < \infty$. Note that the orthogonality of the carrier functions $\{\phi_k\}_{k\in\mathcal{I}}$ was essential for the above calculations.

If for a given $a \in \ell^2(\mathcal{K})$ we have (7), then the corresponding transmit signal s satisfies $\mathrm{PAPR}(s) < \infty$. And conversely, if for a given $a \in \ell^2(\mathcal{K})$ we can find a $b \in \ell^2(\mathcal{K}^{\complement})$ such that the transmit signal s satisfies $\mathrm{PAPR}(s) < \infty$, then we have (7). Thus, the PAPR problem is weakly solvable if and only if for every information sequence $a \in \ell^2(\mathcal{K})$ there exists a compensation signal B such that the transmit signal s, as defined in (8), satisfies $\mathrm{PAPR}(s) < \infty$.

Remark 2. For a given $a \in \ell^2(\mathcal{K})$, the set of $b \in \ell^2(\mathcal{K}^{\complement})$ such that the norm in (7) is finite, is a convex set. Thus, finding the infimum in (7) is in fact a convex optimization problem, as soon as the set of $b \in \ell^2(\mathcal{K}^{\complement})$ that make the norm in (7) finite, is known. We will discuss this point in more detail in Section VI.

Clearly, strong solvability always implies weak solvability. In [18] the question was raised if maybe the converse implication is also true, that is, if maybe both concepts are equivalent. In [19] this equivalence was proved for OFDM by showing that weak solvability implies strong solvability. The question remained whether it is also true for other ONS. The goal of this work is to prove the equivalence of strong solvability and weak solvability for general complete ONS. Since the proof in [19] was tailored to the specific properties of the OFDM ONS, we need to use a completely different approach here.

If for a given ONS $\{\phi_k\}_{k\in\mathcal{I}}$ in $L^2(\mu)$ and set $\mathcal{K} \subset \mathcal{I}$ the PAPR problem is not weakly solvable, there exists an information sequence $a \in \ell^2(\mathcal{K})$ such that the peak value of the transmit signal cannot be bounded, i.e., (7) does not hold. In Section VI we will study the structure of the set of information sequences a for which (7) does not hold, and will see that if the PAPR problem is not weakly solvable, "almost all" information sequences lead to an unbounded signal s.

V. Equivalence of Solvability Concepts

The goal of this section is to show that for arbitrary complete ONS, weak solvability, as stated in Definition 2, implies strong solvability, as stated in Definition 1. Hence, both concepts of stability are equivalent.

To this end, we start with the following simple lemma, which gives a different but equivalent characterization of the weak solvability concept.

Lemma 1. *Let $\{\phi_n\}_{n\in\mathbb{N}}$ be an ONS in $L^2(\mu)$ and $\mathcal{K} \subset \mathbb{N}$. The PAPR problem is weakly solvable for $\{\phi_n\}_{n\in\mathbb{N}}$ and $\mathcal{K}$ if and only if for all $a \in \ell^2(\mathcal{K})$ there exists a $f_a \in L^\infty(\mu)$ such that*

$$\int_0^1 f_a\overline{\phi_k}\,\mathrm{d}\mu = a_k$$

for all $k \in \mathcal{K}$.

Proof. "$\Rightarrow$": Assume that the PAPR problem is weakly solvable for $\{\phi_n\}_{n\in\mathbb{N}}$ and $\mathcal{K}$. Then we have

$$\inf_{b\in\ell^2(\mathcal{K}^{\complement})} \left\| \sum_{k\in\mathcal{K}} a_k\phi_k + \sum_{k\in\mathcal{K}^{\complement}} b_k\phi_k \right\|_{L^\infty(\mu)} < \infty.$$

It follows that there exists a $b^* \in \ell^2(\mathcal{K}^{\complement})$ such that for

$$f_a = \sum_{k\in\mathcal{K}} a_k\phi_k + \sum_{k\in\mathcal{K}^{\complement}} b^*_k\phi_k,$$

where the convergence of the sums is in the $L^2(\mu)$ norm, we have $\|f_a\|_{L^\infty(\mu)} < \infty$. Further, since $\{\phi_n\}_{n\in\mathbb{N}}$ is an ONS, we have

$$\int_\Omega f_a\overline{\phi_k}\,\mathrm{d}\mu = a_k$$

for all $k \in \mathcal{K}$.

"$\Leftarrow$": Let $a \in \ell^2(\mathcal{K})$ be arbitrary but fixed. According to the assumption, there exists a $f_a \in L^\infty(\mu)$ such that

$$\int_\Omega f_a\overline{\phi_k}\,\mathrm{d}\mu = a_k$$

for all $k \in \mathcal{K}$. Since $f_a \in L^\infty(\mu) \subset L^2(\mu)$, the series expansion

$$\sum_{n\in\mathbb{N}} c_n\phi_n \tag{10}$$

with $\{c_n\}_{n\in\mathbb{N}} \in \ell^2$ converges to f_a in the $L^2(\mu)$ norm. Since the convergence of the series (10) is unconditional, the reordering

$$\sum_{k\in\mathcal{K}} a_k\phi_k + \sum_{k\in\mathcal{K}^{\complement}} b_k\phi_k$$

with $a_k = c_k$ for $k \in \mathcal{K}$ and $b_k = c_k$ for $k \in \mathcal{K}^\complement$ also converges to f_a. Since $f_a \in L^\infty(\mu)$, this implies that

$$\left\| \sum_{k\in\mathcal{K}} a_k\phi_k + \sum_{k\in\mathcal{K}^\complement} b_k\phi_k \right\|_{L^\infty(\mu)} = \|f_a\|_{L^\infty(\mu)} < \infty,$$

and it follows that the PAPR problem is weakly solvable for $\{\phi_n\}_{n\in\mathbb{N}}$ and $\mathcal{K}$. □

In the next theorem our main result is presented.

Theorem 2. *Let $\{\phi_n\}_{n\in\mathbb{N}}$ be a complete ONS in $L^2(\mu)$ with $\sup_{n\in\mathbb{N}}\|\phi_n\|_\infty < \infty$, and $\mathcal{K} \subset \mathbb{N}$, such that the PAPR problem is weakly solvable. Then the PAPR problem is strongly solvable, i.e., there exists a constant $C_{EX} = C_{EX}(\mathcal{K}, \{\phi_n\}_{n\in\mathbb{N}})$ such that for all $a \in \ell^2(\mathcal{K})$ we can find a $b \in \ell^2(\mathcal{K}^\complement)$ such that*

$$\left\| \sum_{n\in\mathcal{K}} a_n\phi_n + \sum_{n\in\mathcal{K}^\complement} b_n\phi_n \right\|_{L^\infty(\mu)} \le C_{EX}\|a\|_{\ell^2(\mathcal{K})}.$$

For the proof of Theorem 2 we need the following lemma and the set

$$\mathcal{M}(\mathcal{K}) = \left\{ f \in L^\infty(\mu) : \int_\Omega f\overline{\phi_n}\,\mathrm{d}\mu = 0\ \ \forall n \in \mathcal{K} \right\}.$$

Lemma 2. *$\mathcal{M}(\mathcal{K})$ is a closed subspace of $L^\infty(\mu)$.*

Proof. Clearly, $\mathcal{M}(\mathcal{K})$ has a linear structure, i.e., is closed with respect to addition and multiplication with complex scalars. It remains to prove that $\mathcal{M}(\mathcal{K})$ is closed. Let $\{f_m\}_{m\in\mathbb{N}}$ be an arbitrary sequence in $\mathcal{M}(\mathcal{K})$ that converges in $L^\infty(\mu)$. That is, there exists a $f_* \in L^\infty(\mu)$ such that $\lim_{m\to\infty}\|f_* - f_m\|_{L^\infty(\mu)} = 0$. We need to show that $f_* \in \mathcal{M}(\mathcal{K})$. For $m \in \mathbb{N}$, $n \in \mathcal{K}$ we have

$$\begin{aligned}\left|\int_\Omega f_*\overline{\phi_n}\,\mathrm{d}\mu\right| &= \left|\int_\Omega f_*\overline{\phi_n}\,\mathrm{d}\mu - \int_\Omega f_m\overline{\phi_n}\,\mathrm{d}\mu\right| \\ &= \left|\int_\Omega (f_* - f_m)\overline{\phi_n}\,\mathrm{d}\mu\right| \\ &\le \|\phi_n\|_{L^\infty(\mu)}\|f_* - f_m\|_{L^\infty(\mu)}.\end{aligned}$$

Letting m go to infinity, we see that

$$\left|\int_\Omega f_*\overline{\phi_n}\,\mathrm{d}\mu\right| = 0$$

for all $n \in \mathcal{K}$, which implies that $f_* \in \mathcal{M}(\mathcal{K})$. □

In the proof of Theorem 2 we also employ the bounded inverse theorem, which is a consequence of the open mapping theorem [26, pp. 99]. We state the bounded inverse theorem next for convenience.

Theorem 3 (Bounded Inverse Theorem)**.** *Let B_1, B_2 be two Banach spaces. If $T\colon B_1 \to B_2$ is a bounded linear operator which is also bijective then the inverse operator $T^{-1}\colon B_2 \to B_1$ is bounded as well.*

Now we are in the position to prove Theorem 2.

Proof of Theorem 2. For $f \in L^\infty(\mu)$ we define the set

$$[f] = \left\{ g \in L^\infty(\mu) : \int_\Omega (f-g)\overline{\phi_n}\,\mathrm{d}\mu = 0\ \forall n \in \mathcal{K} \right\}.$$

Let $\mathcal{Q}_\mathcal{K}$ denote the quotient set $L^\infty(\mu)/\mathcal{M}(\mathcal{K})$, consisting of all the sets $[f]$, $f \in L^\infty(\mu)$. $\mathcal{Q}_\mathcal{K}$ has a linear structure: we have $\alpha[f] = [\alpha f]$ and $[f] + [g] = [f+g]$. Further,

$$\|[f]\|_{\mathcal{Q}_\mathcal{K}} = \inf_{g\in\mathcal{M}(\mathcal{K})}\|f+g\|_{L^\infty(\mu)}$$

defines a norm on $\mathcal{Q}_\mathcal{K}$. Equipped with this norm $\mathcal{Q}_\mathcal{K}$ becomes a Banach space.

Next, we consider the operator $R_\mathcal{K}\colon \mathcal{Q}_\mathcal{K} \to \ell^2(\mathcal{K})$, defined by

$$(R_\mathcal{K}[f])(k) = \int_\Omega f\overline{\phi_k}\,\mathrm{d}\mu, \quad k \in \mathcal{K}.$$

For $r \in \mathcal{M}(\mathcal{K})$ we have

$$\begin{aligned}\|R_\mathcal{K}[f]\|_{\ell^2(\mathcal{K})} &= \left(\sum_{k\in\mathcal{K}}\left|\int_\Omega f\overline{\phi_k}\,\mathrm{d}\mu\right|^2\right)^{\frac12} \\ &= \left(\sum_{k\in\mathcal{K}}\left|\int_\Omega (f+r)\overline{\phi_k}\,\mathrm{d}\mu\right|^2\right)^{\frac12} \\ &\le \left(\sum_{k\in\mathbb{N}}\left|\int_\Omega (f+r)\overline{\phi_k}\,\mathrm{d}\mu\right|^2\right)^{\frac12} \\ &= \left(\int_\Omega |f+r|^2\,\mathrm{d}\mu\right)^{\frac12} \\ &\le \|f+r\|_{L^\infty(\mu)}, \end{aligned} \tag{11}$$

where we used Parseval's equality in the second to last line and Hölder's inequality in the last line. Since the left hand side of (11) does not depend on r, it follows that

$$\begin{aligned}\|R_\mathcal{K}[f]\|_{\ell^2(\mathcal{K})} &\le \inf_{r\in\mathcal{M}(\mathcal{K})}\|f+r\|_{L^\infty(\mu)} \\ &= \|[f]\|_{\mathcal{Q}_\mathcal{K}}.\end{aligned}$$

This shows that the operator $R_\mathcal{K}\colon \mathcal{Q}_\mathcal{K} \to \ell^2(\mathcal{K})$ is well-defined and bounded. Further, from $R_\mathcal{K}([f_1] + [f_2]) = R_\mathcal{K}[f_1] + R_\mathcal{K}[f_2]$, $[f_1], [f_2] \in \mathcal{Q}_\mathcal{K}$, and $R_\mathcal{K}(\alpha[f]) = \alpha R_\mathcal{K}[f]$, $\alpha \in \mathbb{C}$, $[f] \in \mathcal{Q}_\mathcal{K}$, we see that $R_\mathcal{K}$ is a linear operator.

Let $[f_1], [f_2] \in \mathcal{Q}_\mathcal{K}$ be arbitrary such that $R_\mathcal{K}[f_1] = R_\mathcal{K}[f_2]$. It follows that

$$(R_\mathcal{K}[f_1])(k) - (R_\mathcal{K}[f_2])(k) = \int_\Omega (f_1 - f_2)\overline{\phi_k}\,\mathrm{d}\mu = 0$$

for all $k \in \mathcal{K}$. Since $\{\phi_n\}_{n\in\mathbb{N}}$ is a complete ONS, this shows that $f_1 = f_2$ μ-a.e., which in turn implies that $[f_1] = [f_2]$. Hence $R_\mathcal{K}$ injective.

Since, according to the assumptions of the theorem, the PAPR problem is weakly solvable, we have due to Lemma 1 that for every $a \in \ell^2(\mathcal{K})$ there exists an $f_a \in L^\infty(\mu)$ such that

$$\int_\Omega f_a\overline{\phi_k}\,\mathrm{d}\mu = a_k, \quad k \in \mathcal{K}.$$

Hence, there exists a $[f_a] \in \mathcal{Q}_\mathcal{K}$ such that $R_\mathcal{K}[f_a] = a$. Since $a \in \ell^2(\mathcal{K})$ was arbitrary, we see that $R_\mathcal{K}\mathcal{Q}_\mathcal{K} = \ell^2(\mathcal{K})$. That is $R_\mathcal{K}$ is also surjective.

We established the fact that $R_\mathcal{K}\colon \mathcal{Q}_\mathcal{K} \to \ell^2(\mathcal{K})$ is a bijective bounded linear operator. As a consequence of Theorem 3, there exists a bounded linear operator $E_\mathcal{K}$ such that $E_\mathcal{K} = R_\mathcal{K}^{-1}$.

Let $\epsilon > 0$ and $a \in \ell^2(\mathcal{K})$ be arbitrary. Then we have $E_{\mathcal{K}}(a) = [f_a]$ and

$$\|[f_a]\|_{\mathcal{Q}_{\mathcal{K}}} \leq \|E_{\mathcal{K}}\|_{\ell^2(\mathcal{K})\to\mathcal{Q}_{\mathcal{K}}} \|a\|_{\ell^2(\mathcal{K})}. \tag{12}$$

Further, according to the definition of $\|\cdot\|_{\mathcal{Q}_{\mathcal{K}}}$, there exists a $g_{\epsilon,a} \in [f_a]$ such that

$$\|g_{\epsilon,a}\|_{L^\infty(\mu)} \leq \|[f_a]\|_{\mathcal{Q}_{\mathcal{K}}} + \epsilon\|a\|_{\ell^2(\mathcal{K})}. \tag{13}$$

Since $g_{\epsilon,a} \in [f_a]$ we have

$$\int_\Omega g_{\epsilon,a}\overline{\phi_k}\,\mathrm{d}\mu = a_k, \quad k \in \mathcal{K}.$$

Combining (12) and (13), we see that

$$\|g_{\epsilon,a}\|_{L^\infty(\mu)} \leq (\|E_{\mathcal{K}}\|_{\ell^2(\mathcal{K})\to\mathcal{Q}_{\mathcal{K}}} + \epsilon)\|a\|_{\ell^2(\mathcal{K})}.$$

Hence, for all $\epsilon > 0$, the PAPR problem is strongly solvable with extension constant $\|E_{\mathcal{K}}\|_{\ell^2(\mathcal{K})\to\mathcal{Q}_{\mathcal{K}}} + \epsilon$. According to Theorem 1, it follows that for all $f \in \mathfrak{F}^1(\mathcal{K})$ and all $\epsilon > 0$ we have

$$\|f\|_{L^2(\mu)} \leq (\|E_{\mathcal{K}}\|_{\ell^2(\mathcal{K})\to\mathcal{Q}_{\mathcal{K}}} + \epsilon)\|f\|_{L^1(\mu)}.$$

Taking the limit $\epsilon \to 0$ on both sides of the inequality yields

$$\|f\|_{L^2(\mu)} \leq \|E_{\mathcal{K}}\|_{\ell^2(\mathcal{K})\to\mathcal{Q}_{\mathcal{K}}}\|f\|_{L^1(\mu)}$$

for all $f \in \mathfrak{F}^1(\mathcal{K})$. Employing Theorem 1 again, we see that the PAPR problem is strongly solvable even with extension constant $\|E_{\mathcal{K}}\|_{\ell^2(\mathcal{K})\to\mathcal{Q}_{\mathcal{K}}}$. □

VI. Application

Next, we apply the theory from the previous sections. We analyze the question what happens if, for a given complete ONS $\{\phi_n\}_{n\in\mathbb{N}}$ and set $\mathcal{K} \subset \mathbb{N}$, the PAPR problem is not strongly solvable for any constant C_{EX}. In this case, the PAPR problem is also not weakly solvable, according to Theorem 2.

For a given complete ONS $\{\phi_n\}_{n\in\mathbb{N}}$ and set $\mathcal{K} \subset \mathbb{N}$, let

$$\mathcal{B}(a) = \left\{ b \in \ell^2(\mathcal{K}^{\complement}) \colon \left\| \sum_{n\in\mathcal{K}} a_n\phi_n + \sum_{n\in\mathcal{K}^{\complement}} b_n\phi_n \right\|_{L^\infty(\mu)} < \infty \right\}.$$

We have $\mathcal{B}(a) = \emptyset$ if and only if the PAPR problem is not weakly solvable. In [19] the set

$$\mathcal{D} = \{a \in \ell^2(\mathcal{K}) \colon \mathcal{B}(a) = \emptyset\},$$

which contains all information sequences $a \in \ell^2(\mathcal{K})$ for which the PAPR problem is not weakly solvable, was analyzed for OFDM. It was shown that if for $\{\mathrm{e}^{i2\pi k\,\cdot}\}_{k\in\mathbb{Z}}$ and $\mathcal{K} \subset \mathbb{Z}$ the PAPR problem is not weakly solvable, then the set $\mathcal{D}$ is a residual set, i.e., is big in a topological sense. Thanks to Theorem 2 we can extend this result to general complete ONSs.

Remark 3. $\mathcal{B}(a)$ is convex set. If $\mathcal{B}(a) \neq \emptyset$ then we have for arbitrary $b_1, b_2 \in \mathcal{B}(a)$ that $b_\lambda = (1-\lambda)b_1 + \lambda b_2 \in \ell^2(\mathcal{K}^{\complement})$. Further, we have

$$\begin{aligned}
&\left\| \sum_{n\in\mathcal{K}} a_n\phi_n + \sum_{n\in\mathcal{K}^{\complement}} b_{\lambda,n}\phi_n \right\|_{L^\infty(\mu)} \\
&= \left\| \sum_{n\in\mathcal{K}} a_n\phi_n + (1-\lambda)\sum_{n\in\mathcal{K}^{\complement}} b_{1,n}\phi_n + \lambda\sum_{n\in\mathcal{K}^{\complement}} b_{2,n}\phi_n \right\|_{L^\infty(\mu)} \\
&= \left\| (1-\lambda)\left(\sum_{n\in\mathcal{K}} a_n\phi_n + \sum_{n\in\mathcal{K}^{\complement}} b_{1,n}\phi_n \right) \right. \\
&\qquad \left. + \lambda\left(\sum_{n\in\mathcal{K}} a_n\phi_n + \sum_{n\in\mathcal{K}^{\complement}} b_{2,n}\phi_n \right) \right\|_{L^\infty(\mu)} \\
&\leq \left\| (1-\lambda)\left(\sum_{n\in\mathcal{K}} a_n\phi_n + \sum_{n\in\mathcal{K}^{\complement}} b_{1,n}\phi_n \right) \right\|_{L^\infty(\mu)} \\
&\qquad + \left\| \lambda\left(\sum_{n\in\mathcal{K}} a_n\phi_n + \sum_{n\in\mathcal{K}^{\complement}} b_{2,n}\phi_n \right) \right\|_{L^\infty(\mu)} \\
&< \infty,
\end{aligned}$$

which shows that $b_\lambda \in \mathcal{B}(a)$. The convexity of the set $\mathcal{B}(a)$ and the $L^\infty(\mu)$-norm implies that finding the infimum in (7) is in fact a convex optimization problem as soon as $\mathcal{B}(a)$ is given. However, in general $\mathcal{B}(a)$ is unknown.

Theorem 4. *Let $\{\phi_n\}_{n\in\mathbb{N}}$ be a complete ONS in $L^2(\mu)$ and $\mathcal{K} \subset \mathbb{N}$ such that the PAPR problem is not weakly solvable. Then the set*

$$\mathcal{D} = \{a \in \ell^2(\mathcal{K}) \colon \mathcal{B}(a) = \emptyset\}$$

is a residual set.

Proof. For $M \in \mathbb{N}$ let

$$\mathcal{Z}_M = \left\{ a \in \ell^2(\mathcal{K}) \colon \exists f \in L^\infty(\mu), \|f\|_{L^\infty(\mu)} \leq M \right. \\ \left. \text{with } \int_\Omega f\overline{\phi_k}\,\mathrm{d}\mu = a_k, k \in \mathcal{K} \right\}.$$

We have

$$\mathcal{D}^{\complement} = \bigcup_{M\in\mathbb{N}} \mathcal{Z}_M.$$

Assume that the PAPR problem is not weakly solvable. Then there exists an $a \in \ell^2(\mathcal{K})$ such that $\mathcal{B}(a) = \emptyset$. We will show that

$$\mathcal{D}^{\complement} = \{a \in \ell^2(\mathcal{K}) \colon \mathcal{B}(a) \neq \emptyset\}$$

is a set of first category. According to the definition of a residual set, this implies that $\mathcal{D}$ is a residual set.

We prove that, for all $M \in \mathbb{N}$, the set $\mathcal{Z}_M$ is nowhere dense in $\ell^2(\mathcal{K})$. Then it follows that $\mathcal{D}^{\complement}$, as the countable union of nowhere dense sets, is a set of first category.

We do a proof by contradiction: We assume that there exists an $M_0 \in \mathbb{N}$ such that $\mathcal{Z}_{M_0}$ is not nowhere dense, and show that this assumption leads to a contradiction. According to the assumption there exist an $\hat{a} \in \ell^2(\mathcal{K})$ and a $\delta > 0$ such that

$$\mathcal{Z}_{M_0} \cap B_\delta(\hat{a})$$

is dense in $B_\delta(\hat{a})$, where

$$B_\delta(\hat{a}) = \{a \in \ell^2(\mathcal{K})\colon \|a - \hat{a}\|_{\ell^2(\mathcal{K})} < \delta\}$$

denotes the open ball at $\hat{a}$ with radius δ.

Let $a \in B_\delta(\hat{a})$ be arbitrary. Since $\mathcal{Z}_{M_0} \cap B_\delta(\hat{a})$ is dense in $B_\delta(\hat{a})$, there exists a sequence $\{a^{(N)}\}_{N\in\mathbb{N}} \subset \mathcal{Z}_{M_0} \cap B_\delta(\hat{a})$ such that

$$\lim_{N\to\infty} \|a - a^{(N)}\|_{\ell^2(\mathcal{K})} = 0.$$

Further, for every $N \in \mathbb{N}$, there exists an $f_N \in L^\infty(\mu)$ with $\|f_N\|_{L^\infty(\mu)} \leq M_0$ such that

$$\int_\Omega f_N \overline{\phi_k}\, \mathrm{d}\mu = a_k^{(N)}, \quad k \in \mathcal{K}.$$

Recall that $L^1(\mu)$ is a separable Banach space. Hence, the closed unit ball in $L^\infty(\mu)$ is sequentially compact in the weak* topology, according to the Banach–Alaoglu theorem [27, p. 68, Th. 3.17]. It follows that there exists an $f_* \in L^\infty(\mu)$, $\|f_*\|_{L^\infty(\mu)} \leq M_0$ and a subsequence of the natural numbers $\{n_r\}_{r\in\mathbb{N}}$ such that

$$\lim_{r\to\infty} \int_\Omega f_{n_r} g\, \mathrm{d}\mu = \int_\Omega f_* g\, \mathrm{d}\mu$$

for all $g \in L^1(\mu)$. Thus, we have for $k \in \mathcal{K}$ that

$$\begin{aligned} \int_\Omega f_* \overline{\phi_k}\, \mathrm{d}\mu &= \lim_{r\to\infty} \int_\Omega f_{n_r} \overline{\phi_k}\, \mathrm{d}\mu \\ &= \lim_{r\to\infty} a_k^{(N_r)} \\ &= a_k. \end{aligned}$$

Hence, we see that $a \in \mathcal{Z}_{M_0}$. Since $a \in B_\delta(\hat{a})$ was arbitrary, it follows that $B_\delta(\hat{a}) \subset \mathcal{Z}_{M_0}$, which also implies that $\hat{a} \in \mathcal{Z}_{M_0}$.

According to the assumption of the theorem the PAPR is not weakly solvable. Hence, there exists an $\tilde{a} \in \ell^2(\mathcal{K})$ with $\mathcal{B}(\tilde{a}) = \emptyset$. We set

$$\alpha := \hat{a} + \frac{\delta}{2\|\tilde{a}\|_{\ell^2(\mathcal{K})}} \tilde{a} \in B_\delta(\hat{a}) \subset \mathcal{Z}_{M_0}.$$

Hence, there must exist an $f_1 \in L^\infty(\mu)$, $\|f_1\|_{L^\infty(\mu)} \leq M_0$ such that

$$\int_\Omega f_1 \overline{\phi_k}\, \mathrm{d}\mu = \alpha_k, \quad k \in \mathcal{K}.$$

Since $\hat{a} \in \mathcal{Z}_{M_0}$, there exists an $f_2 \in L^\infty(\mu)$, $\|f_2\|_{L^\infty(\mu)} \leq M_0$ such that

$$\int_\Omega f_2 \overline{\phi_k}\, \mathrm{d}\mu = \hat{a}_k, \quad k \in \mathcal{K}.$$

For

$$f_3 := \frac{2\|\tilde{a}\|_{\ell^2(\mathcal{K})}}{\delta}(f_1 - f_2)$$

we have

$$\begin{aligned} \|f_3\|_{L^\infty(\mu)} &= \frac{2\|\tilde{a}\|_{\ell^2(\mathcal{K})}}{\delta} \|f_1 - f_2\|_{L^\infty(\mu)} \\ &\leq \frac{2\|\tilde{a}\|_{\ell^2(\mathcal{K})}}{\delta} (\|f_1\|_{L^\infty(\mu)} + \|f_2\|_{L^\infty(\mu)}) \\ &\leq \frac{4\|\tilde{a}\|_{\ell^2(\mathcal{K})} M_0}{\delta}, \end{aligned}$$

which shows that $f_3 \in L^\infty(\mu)$. For $k \in \mathcal{K}$, we have

$$\int_\Omega f_3 \overline{\phi_k}\, \mathrm{d}\mu = \frac{2}{\delta}(\alpha_k - \hat{a}_k) = \tilde{a}_k, \quad k \in \mathcal{K}.$$

Therefore, we have $\tilde{a} \in \mathcal{Z}_{\tilde{M}}$, where $\tilde{M}$ is the smallest natural number such that $\tilde{M} \geq 4\|\tilde{a}\|_{\ell^2(\mathcal{K})} M_0/\delta$. It follows that $\mathcal{B}(\tilde{a}) \neq \emptyset$, which is a contradiction. □

VII. Conclusion

As discussed in the introduction, central questions in the context of tone reservation are: What is the best possible reduction of the PAPR? And: What is the optimal information set that achieves this reduction? The answers to these questions are difficult to obtain and generally unknown. By proving that strong solvability is equivalent to weak solvability, i.e., that a distinction between both concepts is not needed in general multi waveform transmission schemes employing bounded complete ONSs, we establish a first key result towards answering the above questions.

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